Neutrino Flavor Oscillations Using the Dirac Equation

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Abstract

The theory of neutrino flavor rotations is discussed in terms of wave function solutions to the Dirac equation with a neutrino mass matrix. We give a critical review of the nature of neutrino oscillations.

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I. INTRODUCTION

There is considerable interest in quantum interference patterns in space and time which result from the notion of flavor rotations. These include Kaon oscillations and B-meson oscillations which are thought to arise from quark flavor rotations. Some believe that Neutrino oscillations also exist[1-4], but presently these are merely theoretical. Such neutrino oscillations are thought to arise from a possible flavor rotated Neutrino mass matrix. In all such cases, the quantum interference patterns are in part due to mass splitting, but they are also in part due to diffraction effects which would be present even if the mass splitting were not present.

One should not forget that the invention of quantum mechanics was needed to explain amplitude interference oscillations without any recourse to the notion of mass splitting. For example, free electrons are described by a Dirac wave equation

$$\left(-i\hbar\gamma^{\mu}\partial_{\mu} + mc\right)\psi(x) = 0, \tag{1}$$

where m is the electron mass. Using the usual spinors

$$\left(\gamma^{\mu}p_{\mu} + mc\right)u(p) = 0, \quad u^{\dagger}(p)u(p) = 1, \quad p^2 = -(mc)^2, \tag{1}$$

one may construct a solution which consists of two plane waves

$$\psi(x) = A_1 u(p_1)e^{ip_1\cdot x/\hbar} + A_2 u(p_2)e^{ip_2\cdot x/\hbar}, \tag{3}$$

and which yields an oscillating current

$$j^{\mu}(x) = ec\overline{\psi}(x)\gamma^{\mu}\psi(x). \tag{4}$$

It is

$$j^{\mu}(x) = ec\left(|A_1|^2\overline{u}(p_1)\gamma^{\mu}u(p_1) + |A_2|^2\overline{u}(p_2)\gamma^{\mu}u(p_2)\right) + 2ec\Re e\left(A_2^* A_1\overline{u}(p_2)\gamma^{\mu}u(p_1)e^{ik\cdot x}\right), \tag{5}$$

where $\hbar k = p_1 - p_2$. Note that the space-time phase interference factor described by $\exp(ik\cdot x)$ is Lorentz invariant, and that electron interference patterns in space and time are simply

and covariantly described by the current four vector $j^\mu(x)$. Although low energy electron diffraction (LEED) experiments carried out every day[5,6] are most often described in the laboratory rest frame, the description of the observed space-time oscillation in the current $j^\mu(x)$ is not overly difficult to describe in any Lorentz frame. Finally, the description of electron interference by normalized wave packets (explicitly including the spin s),

$$\psi(x) = \sum_s \int A(p,s)u(p,s)e^{ip\cdot x/\hbar}d\Upsilon, \quad d\Upsilon = \frac{d^3\mathbf{p}}{\sqrt{|\mathbf{p}|^2 + m^2c^2}}, \quad u^\dagger(p,s')u(p,s) = \delta_{s's}, \tag{6}$$

by no means eliminates the oscillations in the current

$$j^\mu(x) = ec\sum_{s's}\int\int A^*(p',s')A(p,s)\overline{u}(p',s')\gamma^\mu u(p,s)e^{i(p-p')\cdot x/\hbar}d\Upsilon' d\Upsilon. \tag{7}$$

While a sufficiently wide energy distribution in the wave packet will produce a low oscillation visibility, LEED machines routinely produce an electron beam which allows for the easy observation of electron diffraction via the factor $\exp(i(p-p')\cdot x/\hbar)$ in Eq.(7).

We have reviewed these well known features of electron diffraction oscillations mainly because the case of massive neutrinos has for the most part (in the literature) been treated by a completely different set of rules than those implied by the Dirac equation. Since we feel strongly that the Dirac equation is perfectly adequate to the task of describing freely moving spin 1/2 particles, including massive neutrinos, we wish to compare the Dirac theory to those other theories which rely on a more obscure formalism.

II. DIRAC EQUATION

In the Dirac theory, the massive neutrino wave functions may be denoted by $n_a(x)$ where $a = e, \mu, \tau$ denotes the flavor index and m_a are the neutrino masses. The Dirac equation reads

$$-i\hbar\gamma^\mu\partial_\mu \begin{pmatrix} n_e(x) \\ n_\mu(x) \\ n_\tau(x) \end{pmatrix} + \begin{pmatrix} m_e c & 0 & 0 \\ 0 & m_\mu c & 0 \\ 0 & 0 & m_\tau c \end{pmatrix} \begin{pmatrix} n_e(x) \\ n_\mu(x) \\ n_\tau(x) \end{pmatrix} = 0. \tag{8}$$

where the wave function has twelve components, i.e. four spinor components times three flavor components. The physical neutrino wave function $\nu(x)$ of three possible flavors is produced by the rotation

$$\nu(x)=\begin{pmatrix}\nu_e(x)\\ \nu_\mu(x)\\ \nu_\tau(x)\end{pmatrix}=\begin{pmatrix}R_{ee} & R_{e\mu} & R_{e\tau}\\ R_{\mu e} & R_{\mu\mu} & R_{\mu\tau}\\ R_{\tau e} & R_{\tau\mu} & R_{\tau\tau}\end{pmatrix}\begin{pmatrix}n_e(x)\\ n_\mu(x)\\ n_\tau(x)\end{pmatrix}, \tag{9}$$

and obeys

$$(-i\hbar\gamma^\mu\partial_\mu+\mathcal{M}c)\,\nu(x)=0, \tag{10}$$

where the neutrino mass matrix obeys

$$\mathcal{M}_{ab}=\sum_{c=e,\mu,\tau}R_{ac}m_cR^{-1}_{cb},\quad a,b=(e,\mu,\tau), \tag{11}$$

and the matrix R is unitary

$$R^\dagger=R^{-1}. \tag{12}$$

The general solution of the massive neutrino Dirac equation is best discussed in terms of the propagator.

III. NEUTRINO PROPAGATOR

The neutrino propagator is a twelve by twelve matrix obeying

$$\left(-i\gamma^\mu\partial_\mu+\frac{\mathcal{M}c}{\hbar}\right)G(x-x')=\delta(x-x'), \tag{13}$$

or equally well

$$\left(i\partial'_\mu G(x-x')\gamma^\mu+G(x-x')\frac{\mathcal{M}c}{\hbar}\right)=\delta(x-x'). \tag{14}$$

Consider a space-time region Ω. From the obvious identity

$$\nu(x)=\int_\Omega\delta(x-x')\nu(x')d^4x',\ \ x\in\Omega, \tag{15}$$

and Eq.(14) one finds

$$\nu(x)=\int_\Omega\left(i\partial'_\mu G(x-x')\gamma^\mu+G(x-x')\frac{\mathcal{M}c}{\hbar}\right)\nu(x')d^4x',\ \ x\in\Omega, \tag{16}$$

and

$$\nu(x) = \int_\Omega \left[\left(i\partial'_\mu G(x-x')\gamma^\mu \right) \nu(x') + G(x-x') \left(i\gamma^\mu \partial'_\mu \nu(x') \right) \right] d^4x', \;\; x \in \Omega, \tag{17}$$

where Eq.(10) has been employed. Eq.(17) implies

$$\nu(x) = i \int_\Omega \partial'_\mu \left(G(x-x')\gamma^\mu \nu(x') \right) d^4x', \;\; x \in \Omega, \tag{18}$$

which may be converted into a "three surface" integral on the boundary $\partial\Omega$ of the region Ω, i.e.

$$\nu(x) = i \oint_{\partial\Omega} G(x-x')\gamma^\mu \nu(x') d^3\Sigma'_\mu, \;\; x \in \Omega. \tag{19}$$

From Eq.(19) it is evident that the propagator allows one to compute the full neutrino wave function for all $x \in \Omega$, if the wave function is known on the boundary $x' \in \partial\Omega$. Writing Eq.(19) with flavor indices made explicit yields

$$\nu_a(x) = i \sum_{b=e,\mu,\tau} \oint_{\partial\Omega} G_{ab}(x-x')\gamma^\mu \nu_b(x') d^3\Sigma'_\mu, \;\; x \in \Omega, \;\; a = (e,\mu,\tau), \tag{20}$$

where

$$G_{ab}(x-x') = \sum_{c=e,\mu,\tau} R_{ac} R^{-1}_{cb} S(x-x'; m_c), \tag{21}$$

and $S(x-x';m)$ is the ordinary Dirac-Feynman propagator

$$\left(-i\gamma^\mu \partial_\mu + \frac{mc}{\hbar} \right) S(x-x';m) = \delta(x-x'). \tag{22}$$

The complete solution to the neutrino wave function problem is then formally

$$\nu_a(x) = i \sum_{b=e,\mu,\tau} \sum_{c=e,\mu,\tau} \oint_{\partial\Omega} R_{ac} S(x-x';m_c)\gamma^\mu R^{-1}_{cb} \nu_b(x') d^3\Sigma'_\mu, \;\; x \in \Omega. \tag{23}$$

Eq.(13) can be solved[7] in the form

$$G(x-x') = \left(i\gamma^\mu \partial_\mu + \frac{\mathcal{M}c}{\hbar} \right) \mathcal{D}(x-x'), \tag{24}$$

where

$$\left(-\partial_\mu \partial^\mu + (\mathcal{M}c/\hbar)^2 \right) \mathcal{D}(x-x') = \delta(x-x'). \tag{25}$$

From Eq.(19) and (24) it follows that

$$\nu(x) = \left(i\gamma^\mu \partial_\mu + \frac{\mathcal{M}c}{\hbar} \right) \varphi(x), \tag{26}$$

where

$$\varphi(x) = i \oint_{\partial\Omega} \mathcal{D}(x - x')\gamma^\mu \nu(x') d^3\Sigma'_\mu, \quad x \in \Omega. \tag{27}$$

Eq.(27) is a proper starting point for discussing possible neutrino oscillation experiments.

IV. OUTGOING NEUTRINO WAVES

Here we consider neutrinos (going forward in time) in a fixed Lorentz frame. Anti-neutrinos (going backward in time) may be discussed similarly. Let $\nu(\mathbf{r}, 0)$ be the initial neutrino wave-packet function at time zero in the Lorentz frame of interest. At later times, Eq.(27) reads (with $\beta \equiv \gamma^0$)

$$\varphi(\mathbf{r}, t) = i \int \mathcal{D}(\mathbf{r} - \mathbf{r}', t)\beta\nu(\mathbf{r}', 0) d^3\mathbf{r}', \quad (t > 0). \tag{28}$$

From Eq.(25)

$$\mathcal{D}(\mathbf{r} - \mathbf{r}', t) = \int_{-\infty}^{\infty} \left(\frac{e^{i\mathcal{P}(E)|\mathbf{r}-\mathbf{r}'|/\hbar}}{4\pi|\mathbf{r} - \mathbf{r}'|} \right) e^{-iEt/\hbar} \left(\frac{dE}{2\pi\hbar c} \right) \tag{29}$$

where the flavor matrix

$$c\mathcal{P}(E) = \sqrt{E^2 - (\mathcal{M}c^2)^2}. \tag{30}$$

Putting

$$\varphi(\mathbf{r}, t) = \int_{-\infty}^{\infty} \varphi_E(\mathbf{r}) e^{-iEt/\hbar} \left(\frac{dE}{2\pi\hbar c} \right), \tag{31}$$

one finds

$$\varphi_E(\mathbf{r}) = \int \left(\frac{ie^{i\mathcal{P}(E)|\mathbf{r}-\mathbf{r}'|/\hbar}}{4\pi|\mathbf{r} - \mathbf{r}'|} \right) \beta\nu(\mathbf{r}', 0) d^3\mathbf{r}'. \tag{32}$$

For large distances

$$\varphi_E(\mathbf{r}) = \left(\frac{e^{i\mathcal{P}(E)|\mathbf{r}|/\hbar}}{|\mathbf{r}|} \right) f(E, \hat{\mathbf{r}}), \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|}, \quad |\mathbf{r}| >> |\mathbf{r}'|, \tag{33}$$

with the production amplitude (twelve components for spin and flavor)

$$f(E, \hat{\mathbf{r}}) = \left(\frac{i}{4\pi} \right) \int e^{-i\mathcal{P}(E)\hat{\mathbf{r}}\cdot\mathbf{r}'/\hbar} \beta\nu(\mathbf{r}', 0) d^3\mathbf{r}'. \tag{34}$$

In space and time, the asymptotic outgoing wave is

$$\varphi(\mathbf{r},t)=\frac{1}{|\mathbf{r}|}\int_{-\infty}^{\infty}e^{i(\mathcal{P}(E)|\mathbf{r}|-iEt)/\hbar}f(E,\hat{\mathbf{r}})\left(\frac{dE}{2\pi\hbar c}\right). \tag{35}$$

Employing Eq.(11), one finds with flavor indices made explicit

$$\varphi_a(\mathbf{r},t)=\frac{1}{|\mathbf{r}|}\sum_{b=e,\mu,\tau}\sum_{c=e,\mu,\tau}R_{ab}R^{-1}_{bc}\int_{-\infty}^{\infty}e^{i(p_b(E)|\mathbf{r}|-iEt)/\hbar}f_c(E,\hat{\mathbf{r}})\left(\frac{dE}{2\pi\hbar c}\right). \tag{36}$$

where

$$cp_b(E)=\sqrt{E^2-(m_bc^2)^2}. \tag{37}$$

Evaluating the energy integrals on the right hand side of Eq.(36) by steepest descents, one finds the energy

$$\frac{\partial}{\partial E}\left(p_b(E)|\mathbf{r}|-Et\right)_{E=E_b}=0, \tag{38}$$

yielding

$$\frac{|\mathbf{r}|}{t}\equiv v_b,\quad E_b=\frac{m_bc^2}{\sqrt{1-(v_b/c)^2}},\quad p_b(E_b)=\frac{m_bv_b}{\sqrt{1-(v_b/c)^2}}, \tag{39}$$

so with the proper time defined by

$$\tau_b=t\sqrt{1-(v_b/c)^2}=-\left(\frac{p_b(E_b)|\mathbf{r}|-E_bt}{c^2m_b}\right), \tag{40}$$

Eq.(37) reads (putting the integration variable $E=E_b+\varepsilon$),

$$\varphi_a(\mathbf{r},t)\approx\frac{1}{|\mathbf{r}|}\sum_{b=e,\mu,\tau}\sum_{c=e,\mu,\tau}R_{ab}e^{-ic^2m_b\tau_b/\hbar}R^{-1}_{bc}F_c\left(E_b,t-\frac{|\mathbf{r}|}{v_b},\hat{\mathbf{r}}\right). \tag{41}$$

where

$$F_c(E_b,t,\hat{\mathbf{r}})=\int_{-\infty}^{\infty}e^{-i\varepsilon t/\hbar}f_c(E_b+\varepsilon,\hat{\mathbf{r}})\left(\frac{d\varepsilon}{2\pi\hbar c}\right). \tag{42}$$

Eq.(41) is the central result of this work.

Eq.(41) contains the phase factors $\exp(-ic^2m_b\tau_b/\hbar)$, which are determined by the phase velocities, and the group envelope $F_c\left(E_b,t-(|\mathbf{r}|/v_b),\hat{\mathbf{r}}\right)$which propagates with the group velocity. This decomposition (phase velocity in the phase and group velocity in the envelope) has been standard quantum mechanics for well over half a century. See for example the standard scattering theory treatise of Goldberger and Watson[8]. We stress this point because in the literature some workers put the group velocity into the phase, thus forgetting why the phrase "phase velocity" exists.

V. EXPERIMENTAL IMPLICATIONS

In experiments, there is a total outgoing amplitude (for a neutrino of flavor a, when the initial neutrino had flavor c) given by Eq.(41); It is

$$\mathcal{F}^{total}_{c\to a} = \left[\sum_{b=e,\mu,\tau} R_{ab} e^{-ic^2 m_b \tau_b/\hbar} R^*_{cb} F_c \left(E_b, t - \frac{|\mathbf{r}|}{v_b}, \hat{\mathbf{r}} \right) \right]. \tag{43}$$

The absolute value squared of the amplitude is then

$$\left| \mathcal{F}^{total}_{c\to a} \right|^2 = \left[\sum_{b=e,\mu,\tau} \sum_{d=e,\mu,\tau} e^{i\phi_{bd}} R_{ab} R^*_{cb} R^*_{ad} R_{cd} F_c \left(E_b, t - \frac{|\mathbf{r}|}{v_b}, \hat{\mathbf{r}} \right) F_c \left(E_d, t - \frac{|\mathbf{r}|}{v_d}, \hat{\mathbf{r}} \right) \right], \tag{44}$$

with the neutrino oscillation phase factors

$$\phi_{bd} = (m_d c^2 \tau_d - m_b c^2 \tau_b)/\hbar. \tag{45}$$

The neutrino kinematics in the (laboratory frame) regime $m_b c^2 \ll E_b$ must now be discussed. The velocities appearing in Eq.(43) are given by

$$v_b = c\sqrt{1 - \frac{m_b^2 c^4}{E_b^2}} \approx c\left(1 - \frac{m_b^2 c^4}{2E_b^2} + \dots\right), \ (m_b c^2 \ll E_b), \tag{46}$$

very slightly less than light velocity. For example, if $(m_b c^2/e) < 1volt$, and $(E_b/e) > 10^6 volt$, then $[(c - v_b)/c] < 5 \times 10^{-13}$; i.e. the velocity (laboratory frame) is less than light velocity only by the thirteenth significant figure. In the CHORUS and NOMAD experiments where neutrino energies are much higher ($\sim Gev$) than the above estimates, it is safe to ignore the differences in the wave-packets F_c in Eq.(44) and use the phase coherent flavor conversion transition probability

$$\mathcal{P}_{coherent}(c \to a) = \sum_{b=e,\mu,\tau} \sum_{d=e,\mu,\tau} R_{ab} R^*_{cb} R^*_{ad} R_{cd} \exp\left(\frac{i(m_d c^2 \tau_d - m_b c^2 \tau_b)}{\hbar} \right), \tag{47}$$

where Eq.(45) has been employed. Note that the use of many proper times in Eq.(47) (i.e. the fact that each of the neutrinos with mass $m_{b,d}$ has its own internal proper time $\tau_{b,d}$), makes the phase coherent flavor conversion probability in Eq.(47) Lorentz invariant.

This coherent flavor conversion probability is not so obvious when neutrinos come from distant stars. Intuitively, one would expect that the neutrino wave packet of a heavy massive

neutrino arrive on earth later than the wave packet from a light neutrino, if the original neutrino source was (say) an exploding star. The wave packet peaks for two different mass neutrinos would be separated in space by $\Delta r \sim r(\Delta v/c)$ where Δv is the difference in the two neutrino velocities and r is the distance from the distant star to the earth. This means that

$$\Delta r \sim cm \left[\frac{r}{light - years}\right] \left[\frac{\Delta v}{c}\right] \times 10^{18} \quad (stellar.source). \tag{48}$$

Thus the phase coherence in Eq.(47) would be scrambled and one would expect (when the source is a distant star) an incoherent flavor conversion transition probability

$$\mathcal{P}_{incoherent}(c \to a) = \sum_{b=e,\mu,\tau} |R_{ab}|^2 |R_{cb}|^2, \quad (stellar.source). \tag{49}$$

Here Δr is so very much larger than the neutrino detecting nuclear event length scale in the target on earth.

If the source of neutrinos is the sun, then the wave packet peak separation is given by

$$\Delta r \sim cm \left[\frac{r}{R_{earth-sun}}\right] \left[\frac{\Delta v}{c}\right] \times 10^{13}, \tag{50}$$

where $R_{earth-sun}$ is the distance from the earth to the sun. Again the incoherent flavor conversion transition probability applies;

$$\mathcal{P}_{incoherent}(c \to a) = \sum_{b=e,\mu,\tau} |R_{ab}|^2 |R_{cb}|^2, \quad (sun\, source). \tag{45}$$

Let us now compare the central result of this work, i.e. Eq.(47), to similar results as they appear in the literature[9-11].

VI. PREVIOUS ABUSES OF DIRAC NOTATION

The conventional textbook discussions of neutrino oscillations with flavor mixing are simply described as follows: In Dirac notation, the three flavors of neutrino $(\nu_e, \nu_\mu, \nu_\tau)$ are related to those neutrinos of fixed mass

$$\mathcal{M}|n_a >= M_a|n_a >, \ \ (a = e, \mu, \tau), \tag{46}$$

by a flavor rotation matrix via

$$|\nu_a>=\sum_{b=e,\mu,\tau} R_{ab}|n_b>,\ \ |n_a>=\sum_{b=e,\mu,\tau}(R^{-1})_{ab}|\nu_b>,\ \ (a=e,\mu,\tau). \tag{47}$$

In the course of time, the state of a given flavor neutrino changes via

$$\exp\left(-iHt/\hbar\right)|\nu_c>=\sum_{b=e,\mu,\tau} R_{cb}\exp(-iE_b t/\hbar)|n_b>, \tag{48}$$

where

$$E_b=\sqrt{c^4m_b^2+c^2|\mathbf{p}|^2},\ \ (b=e,\mu,\tau). \tag{49}$$

and where $\mathbf{p}$ is the neutrino three-momentum. Eqs.(47) and (48) imply that

$$<\nu_a|\exp\left(-iHt/\hbar\right)|\nu_c>=\sum_{b=e,\mu,\tau} R_{cb}(R^{-1})_{ba}\exp(-iE_b(\mathbf{p})t/\hbar). \tag{50}$$

One then concludes that the transition probabilities between neutrinos of different flavors,

$$P_{WRONG}(c\to a,t)=|<\nu_a|\exp\left(-iHt/\hbar\right)|\nu_c>|^2\ ????, \tag{51}$$

oscillate in time. The above described conventional arguments are clear, elegant, appealing, and wrong.

One may suspect that the conventional discussion above has some problems with Lorentz invariance by applying it (as very many workers have done) to a neutrino produced in the sun and detected on earth. Suppose that a nucleus "at rest" in the sun fires off a beta-decay neutrino. There will be according to the above (incorrect but conventional) discussion a neutrino superposition of three *different energies* E_b, $(b=e,\mu,\tau)$ but *all with the same three-momenta* $\mathbf{p}$. Not all of the nuclei on the sun are "at rest". Now suppose that a moving nucleus fires off a beta-decay neutrino. Can one believe that the three energies are yet again all different but the three-momenta are all yet again the same? The answer is obviously no! If Lorentz symmetry is invoked, then momentum and energy together form a four vector $p=(\mathbf{p},E/c)$. One should put $p_b^2=-(m_bc)^2$ for an on mass shell neutrino, and which of the four momentum *components* may be the same or may be different will then depend on the reference frame. At least the *possibility* that the neutrino is in a superposition of

different three-momenta should be *considered*. This requires that Eq.(49) for an on mass shell neutrino should be replaced by

$$E_b = \sqrt{c^4 m_b^2 + c^2 |\mathbf{p}_b|^2}, \quad (b = e, \mu, \tau). \tag{52}$$

With this *replacement* the conventional discussion is *now* clear, elegant, appealing, and *still* wrong.

Exactly what is the amplitude $< \nu_{final}| \exp(-iHt/\hbar)|\nu_{initial} >$ supposed to mean? If we assume that the neutrino is a simple spin 1/2 particle without further internal structure (beyond the flavor label), then perhaps it is supposed to mean something like

$$< \nu_{final}| \exp(-iHt/\hbar)|\nu_{initial} >= \int \nu^{\dagger}_{final}(\mathbf{r}) \exp(-iHt/\hbar)\nu_{initial}(\mathbf{r}) d^3 r \ \ ???? \tag{53}$$

We have chosen a time t (which means we have chosen a Lorentz reference frame), and a standard Dirac Hamitonian (with flavor indices implicit for example in the mass matrix $\mathcal{M}$)

$$H = c\alpha \cdot \mathbf{p} + \mathcal{M}c^2\beta, \qquad (\mathbf{p} = -i\hbar\nabla), \tag{54}$$

If the above is indeed the case, then there is a remaining problem. In Eq.(53), the inner product contains an *integral over space*.

Space is not the same as time. When there are energy superpositions there will be time oscillations. When there are momentum superpositions there will be space oscillations. Furthermore, when the neutrino comes from the sun to the earth, we know mainly *where* the neutrino was detected. The predicted oscillations should be (at least in part) in space. An *integration over space* seems a peculiar way to get quantum interference (oscillation) diffraction patterns *that exist in space*. Consider electron diffraction (in real experiments scattering electrons off a crystal face). If "thought experiments" are more appealing, then consider two-slit electron diffraction. To get the interference (oscillation) pattern in space, one absolute squares a spatial amplitude. If you integrate over space, then you get a total probability of unity (which is true but not really of much use in discussing diffraction oscillations).

Our suggestion has been in the previous sections the following: (i) If one wants to study neutrino oscillations in space-time, then it is best to study a wave function which actually depends on space and time; e.g. try the Dirac equation Eq.(10). (ii) If one wants to study how the neutrinos are distributed in space and time, then it is best to study the current $j^\mu(x) = c\overline{\nu}(x)\gamma^\mu\nu(x)$ which actually describes the distribution in space and time. (iii) Finally, if one wants to study how a neutrino can propagate from the sun to the earth (or across a large laboratory), then it is best to study the propagator Eqs.(13) and (19) which actually describes how the neutrino propagates.

VII. CONCLUSION

The coherent neutrino oscillation flavor conversion probability

$$\mathcal{P}_{coherent}(c \to a) = \sum_{b=e,\mu,\tau} \sum_{d=e,\mu,\tau} R_{ab} R^*_{cb} R^*_{ad} R_{cd} \exp\left(\frac{i(m_d c^2 \tau_d - m_b c^2 \tau_b)}{\hbar}\right) \tag{55}$$

depends on the proper times of the massive neutrinos; i.e.

$$\hbar\phi_b = -m_b c^2 \tau_b = (p_b|\mathbf{r}| - E_b t). \tag{56}$$

Some previous workers employ only the energy-time factor $E_b t$ in the phase ϕ_b, while other previous workers employ only the momentum-space factor $p_b|\mathbf{r}|$ in the phase ϕ_b. To consider only energy-time or momentum-space and (not both) tells only half the story. The true Lorentz invariant phase ϕ_b requires both.

Finally, neutrino flavor conversions may be described sometimes by $\mathcal{P}_{coherent}(c \to a)$ and sometimes by $\mathcal{P}_{incoherent}(c \to a)$, depending upon what is assumed about neutrino masses, rotation matrix elements, energies of neutrino sources and length scales of actual experiments. There is no universal simple formula which holds true in all regimes.

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